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New constraint for Black Holes in $\mathcal{N} = 2$, $D = 5$ supergravity with matter

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Abstract

With the general aim to classify BPS solutions in $\mathcal{N} = 2$, $D = 5$ supergravity with hypermultiplets and vector multiplets, here we consider a family of static spacetime metrics containing black hole-like solutions, with generic hypermultiplets coupled to radially symmetric electrostatic vector multiplets. We derive the general conditions which the fields must satisfy and determine the form of the fixed point solutions.

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1 Introduction

In this paper we generalize what it has been done in [1], with the introduction of an arbitrary number of vector multiplets considering only abelian gauge groups $(U(1)^{n_V+1})$. This extension allows to consider new classes of configurations due to the richer structure that characterizes the potential [2]. From a more general point of view this extension can be useful to understand which new features arise when considering charged solutions in presence of both hypermultiplet and vector multiplet couplings. A prominent role in this class is taken by the black hole configurations [3]. They have been in the recent years ([1] and references therein) and continue to be [4] an, apparently never-ending, source of new insights in string theory. For this reason a lot of efforts have been produced in the classification of BPS solutions [5] and in particular of black holes in the ungauged and gauged supergravity coupled with vector multiplets in AdS_5 background [6], [7], [8].

The work is organized as follows. We start presenting the model and the main ingredients of the theory [9]. We derive the integrability conditions for this case using the same ansatz of [1] for the metric and for the gauge fields and we study them together with the hyperini and gaugini equations. As for the flat domain configurations of [2] we obtain that the BPS conditions ensure the stability of the potential as shown in [10], [11] and that the supersymmetric flow equations are controlled by the superpotential W .

Although the set of differential equations we get seems analogous to the sub-case $n_V = 0$ treated in the our previous work, a totally new feature arises. Indeed we find also for the scalars of very special geometry the behavior $\varphi'^\Lambda \propto \partial_\Lambda W$ (where with this notation we indicate generically all the scalars) as in [2],¹ [12].

But in addition the special geometry imposes a sort of consistency constraint on the space-dependence of the scalars of the vector multiplets. The consequences of such constraint are quite relevant: for example in the $n_V = 1$ case this determines completely the shape of the scalar for *any choice* of the gauging. The analysis of this constraint (sec. 4) together with a preliminary study of the fixed point solutions (sec. 3) is a necessary conditions to construct a black hole configuration.

We explicitly show that the BPS conditions satisfy the equations of motion (to not tire the reader the calculations are given in the appendix). At the end we conclude discussing the consequences and the possible applications of our results.

2 The model and its BPS equations

We consider $\mathcal{N}=2$ supergravity in five dimensions with an arbitrary number of hypermultiplets and vector multiplets. The field content of the theory is the following:

- the supergravity multiplet

$$\{e_\mu^a, \psi_\mu^{\alpha i}, A_\mu^0\} \quad (2.1)$$

¹As it will be emphasized later on, the factor of proportionality is not longer the same for vector multiplet and hypermultiplet scalars.

containing the *graviton* e_μ^a , two *gravitini* $\psi_\mu^{\alpha i}$ and the *graviphoton* A_μ^0 ;

- n_H hypermultiplets

$$\{\zeta^A, q^X\} \quad (2.2)$$

containing the *hyperini* ζ^A with $A = 1, 2, \dots, 2n_H$, and the *scalars* q^X with $X = 1, 2, \dots, 4n_H$ which define a quaternionic Kahler manifold with metric g_{XY} ;

- the vector multiplet

$$\{A_\mu^{\hat{I}}, \lambda^{ia}, \phi^x\} \quad (2.3)$$

containing n_V *gaugini* λ^{ia} , $a = 1, \dots, n_V$ with spin $\frac{1}{2}$, n_V real scalars ϕ^x , $x = 1, \dots, n_V$ which define a very special manifold and n_V *gauge vectors* $A_\mu^{\hat{I}}$, $\hat{I} = 1, \dots, n_V$. Usually the graviphoton is included by taking $I = 0 \dots n_V$.

The bosonic sector of the gauged Lagrangian density is given by [9]

$$\begin{aligned} \mathcal{L}_{BOS} = & \frac{1}{2}e\{R - \frac{1}{2}a_{IJ}F_{\mu\nu}^I F^{J\mu\nu} - g_{XY}D_\mu q^X D^\mu q^Y - g_{xy}D_\mu \phi^x D^\mu \phi^y - 2g^2\mathcal{V}(q, \phi)\} \\ & + \frac{1}{6\sqrt{6}}\epsilon^{\mu\nu\rho\sigma\tau}C_{IJK}F_{\mu\nu}^I F_{\rho\sigma}^J A_\tau^K \end{aligned} \quad (2.4)$$

with

$$\begin{aligned} D_\mu q^X &= \partial_\mu q^X + gA_\mu^I K_I^X(q) \\ D_\mu \phi^x &= \partial_\mu \phi^x + gA_\mu^I K_I^x(\phi) \end{aligned}$$

where $K_I^X(q)$, $K_I^x(\phi)$ are the Killing vectors on the quaternionic and the very special real manifold respectively and $\mathcal{V}(q, \phi)$ is the scalar potential as given in Appendix.

At this point we concentrate our attention to the abelian case: this implies that the action of the gauge group is non trivial only on the quaternionic manifold while scalars of vector multiplet are uncharged under it. This means that $D_\mu \phi^x \equiv \partial_\mu \phi^x$ and the existence of any isometry for the very special geometry is not required. Then the variations of the fermions for abelian gauge symmetry $U(1)^{n_V+1}$ reduce to:

for the gravitini

$$\begin{aligned} \delta_\epsilon \psi_{\mu i} &= \partial_\mu \epsilon_i + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon_i - \partial_\mu q^X p_{Xi}{}^j \epsilon_j + gA_\mu^I P_{Ii}{}^j \epsilon_j \\ &+ \frac{i}{4\sqrt{6}}(\gamma_{\mu\nu\rho} - 4g_{\mu\nu}\gamma_\rho)h_I F^{I\nu\rho}\epsilon_i - \frac{i}{\sqrt{6}}gh^I P_{Ii}{}^j \gamma_\mu \epsilon_j = 0 \end{aligned} \quad (2.5)$$

for gaugini

$$\delta_\epsilon \lambda_i^x = [-i\phi^{Ix}e^{-w}\gamma_1\delta_i{}^j - 2igh^{Ix}P_I^s(\sigma_s)_i{}^j]$$

$$+\sqrt{\frac{3}{2}}e^{-w}h_I^x\left(v'a^I+a^{I'}\right)\gamma_{01}\delta_i^j\Big]\epsilon_j=0 \quad (2.6)$$

and for hyperini

$$\delta_\epsilon\zeta^A = f_{iX}^A\left[-iq'^X e^{-w}\gamma_1 + i\sqrt{\frac{3}{2}}ga^IK_I^X\gamma_0 + \sqrt{\frac{3}{2}}gh^IK_I^X\right]\epsilon^i = 0 \quad (2.7)$$

where we have set $\phi'^x := \partial_r\phi^x$ and $q'^X := \partial_rq^X$.²

As already explained at the beginning, we want to consider the direct generalization of the problem considered in [1]. We look for electrostatic spherical solutions that preserve half of the $\mathcal{N} = 2$ supersymmetries. We choose the same metric of the previous paper, which is $SO(4)$ symmetric with all the other fields that only depend on the holographic space-time coordinate r . Moreover we fix the gauge for the gauge fields keeping only the A_t^I component different from zero.

Introducing spherical coordinates $(t, r, \theta, \phi, \psi)$ we write

$$ds^2 = -e^{2v}dt^2 + e^{2w}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2) \quad (2.8)$$

where the functions v and w depend on r only.

We parametrize the vector fields as

$$A_t^I = \sqrt{\frac{3}{2}}a^I(r)e^v \quad (2.9)$$

so that

$$A_t^I = \sqrt{\frac{3}{2}}\left(v'a^I + a^{I'}\right)e^v \quad (2.10)$$

2.1 Integrability conditions

We now consider the BPS equations for the gravitini: their integrability condition is the vanishing of their commutators; using the general formulas in [13] one finds only four independent commutators

$$\begin{aligned} &\left\{\left[-\frac{1}{2}\partial_r(v'e^{v-w}) + \frac{1}{2}e^{v-w}(v'a + a')^2 + \frac{g^2}{2}e^{v+w}W^2\right]\gamma_0\gamma_1\delta_i^j\right. \\ &- \sqrt{\frac{3}{2}}ige^v(v'a^I + a^{I'})P_I^s(\sigma_s)_i^j - \sqrt{\frac{3}{2}}ige^va^I\tilde{D}_rP_I^s(\sigma_s)_i^j - \frac{g}{\sqrt{6}}e^v\tilde{D}_rP^s(\sigma_s)_i^j\gamma_0 \\ &\left.- \left[\frac{i}{2}\partial_r(e^{-w}(v'a + a'))e^v\delta_i^j + g^2e^{v+w}a^Ih^JP_I^rP_J^s\epsilon_{rs}{}^t(\sigma_t)_i^j\right]\gamma_1\right\}\epsilon_j = 0 \quad (2.11) \end{aligned}$$

²This notation applies to all the quantities with the only exception of a^I for which we explicitly define $a' \equiv (\partial_r a^I)h_I$. This choice is motivated by the aim to be manifest the similarities with the case $n_V = 0$.

$$\left\{ \left[-\frac{1}{2}v'e^{v-2w} + \frac{g^2}{2}re^vW^2 \right] \gamma_0 \delta_i^j + \frac{ig}{\sqrt{6}}re^{v-w}(v'a+a')P^s(\sigma_s)_i^j \gamma_1 \right. \\ \left. + \left[\frac{i}{2}e^{v-2w}(v'a+a')\delta_i^j + g^2re^v a^I h^J P_I^r P_J^s \epsilon_{rst}(\sigma^t)_i^j \right] \right\} \epsilon_j = 0 \quad (2.12)$$

$$\left\{ \left[\frac{1}{2}\partial_r(e^{-w}) - \frac{g^2}{2}re^wW^2 \right] \gamma_1 \delta_i^j + \frac{i}{4}\partial_r [re^{-w}(v'a+a')] \gamma_0 \gamma_1 \delta_i^j \right. \\ \left. - \frac{ig}{\sqrt{6}}r(v'a+a')P^s(\sigma_s)_i^j \gamma_0 + \frac{ig}{\sqrt{6}}r\tilde{D}_r P^s(\sigma_s)_i^j \right\} \epsilon_j = 0 \quad (2.13)$$

$$\left\{ \frac{1}{2} \left[1 - e^{-2w} - \frac{r^2 e^{-2w}}{4} (v'a+a')^2 + \frac{2}{3}g^2 r^2 W^2 \right] \delta_i^j + \frac{i}{2}re^{-2w}(v'a+a')\delta_i^j \gamma_0 \right. \\ \left. + \frac{ig}{\sqrt{6}}r^2 e^{-w}(v'a+a')P^s(\sigma_s)_i^j \gamma_0 \gamma_1 \right\} \sin \theta \epsilon_j = 0 \quad (2.14)$$

with the scalar derivative defined as

$$\tilde{D}_\mu = \partial_\mu \phi^x \partial_x + \partial_\mu q^X D_X$$

Here we have defined $a := h_I a^I$ and $a' := h_I a^{I'}$.

2.2 Matter field conditions: Hyperini equation

Now we compare the information coming from the integrability condition with the supersymmetric variation of the matter fermions. We consider first the equation for the hyperini. Assuming that

$$re^{-2w}(v'a+a') \neq 0 \quad (2.15)$$

we can rewrite (2.14) in the form

$$(if^0 \gamma_0 \delta_l^k + f^r (\sigma_r)_l^k \gamma_1) \epsilon_k = \epsilon_l \quad (2.16)$$

with

$$f^r = -gre^w W Q^r \quad (2.17)$$

$$f^0 = -\frac{1 - e^{-2w} - \left(\frac{re^{-w}}{2}(v'a+a') \right)^2 + g^2 r^2 W^2}{re^{-2w}(v'a+a')} \quad (2.18)$$

Also if we define $\Lambda = -f^0$ we have

$$f^r = \pm \sqrt{1 - \Lambda^2} Q^r \quad (2.19)$$

Put in (2.7) and using (A.6) we obtain

$$[iA\delta_l^k + B^s(\sigma_s)_l^k]\gamma_1\epsilon_k = [C\delta_l^k + iD^s(\sigma_s)_l^k]\epsilon_k \quad (2.20)$$

with

$$A = +\frac{1}{2}q'^Z e^{-w} f^0 + \sqrt{\frac{3}{2}}ga^I D^Z P_I^s f_s \quad (2.21)$$

$$B^s = +\frac{1}{2}\sqrt{\frac{3}{2}}ga^I K_I^Z f^s - q'^X R^{sZ}{}_X e^{-w} f^0 - \sqrt{\frac{3}{2}}ga^I D^Z P_r f_t \epsilon^{rts} \quad (2.22)$$

$$C = \frac{1}{2}\sqrt{\frac{3}{2}}a^I K_I^Z g + \frac{1}{2}\sqrt{\frac{3}{2}}f^0 g K^Z \quad (2.23)$$

$$D^s = \sqrt{\frac{3}{2}}ga^I D^Z P_I^s + g\sqrt{\frac{3}{2}}h^I D^Z P_I^s f^0 \quad (2.24)$$

It is now easy to see that this condition is not compatible with (2.16) so that one must put $A = B^s = C = D^s = 0$ that is³

$$q'^Z e^{-w} \Lambda = \sqrt{6}ga^I D^Z P_I^s f_s \quad (2.25)$$

$$q'^X R^{sZ}{}_X e^{-w} \Lambda = -\frac{1}{2}\sqrt{\frac{3}{2}}ga^I K_I^Z f^s + \sqrt{\frac{3}{2}}ga^I D^Z P_{Ir} f_t \epsilon^{rts} \quad (2.26)$$

$$a^I K_I^Z = K^Z \Lambda \quad (2.27)$$

$$a^I D^Z P_I^s = h^I D^Z P_I^s \Lambda \quad (2.28)$$

Using (2.27) and (2.28) in (2.25) and (2.26) we find

$$q'_Z = \pm 3ge^w \sqrt{1 - \Lambda^2} \partial_Z W \quad (2.29)$$

$$q'^X R^{sZ}{}_X e^{-w} = \mp \sqrt{1 - \Lambda^2} \sqrt{\frac{3}{2}}g \left(\frac{1}{2}K^Z Q^s + \sqrt{\frac{3}{2}}W D^Z Q_r Q_t \epsilon^{rst} \right) \quad (2.30)$$

After contraction of (2.30) with K_Z we obtain

$$\sqrt{\frac{3}{2}}g^2 r e^{2w} W Q^r = -2 \frac{q'^X D_X P^r}{|K|^2} \quad (2.31)$$

which gives

$$q'^X D_X Q^r = 0 \quad (2.32)$$

$$ge^w |K|^2 \sqrt{1 - \Lambda^2} = \pm 2q'^X \partial_X W \quad (2.33)$$

³It is possible to obtain immediately the same result applying the general analysis of the hyperini equation in [14].

Also (2.18) and (2.17) can be rewritten as

$$\sqrt{1 - \Lambda^2} = \mp gre^w W \quad (2.34)$$

$$\Lambda = \frac{1 - e^{-2w} - \left(\frac{re^{-w}}{2} (v'a + a') \right)^2 + g^2 r^2 W^2}{re^{-2w} (v'a + a')} \quad (2.35)$$

Using (2.34) in (2.35) we obtain

$$1 = \Lambda^2 e^{-2w} \left[1 + \frac{r}{2\Lambda} (v'a + a') \right]^2 \quad (2.36)$$

If we use (2.33) in (2.30), the last one becomes

$$K^2 q'^X R_{ZX}^s = -q'^X \partial_X W \left(\sqrt{\frac{3}{2}} Q^s K_Z + 3W Q^t D_Z Q^r \epsilon_{tr}{}^s \right) \quad (2.37)$$

Many other relations, which will be useful to check the equations of motion, follow from (2.29), (2.32), (2.33) and (2.37):

$$|K|^2 |q'|^2 = 6 (q'^X \partial_X W)^2 \quad (2.38)$$

$$|q'|^2 K_Z = 2\sqrt{6} \delta_{rs} q'^X R_{XZ}^r Q^s q'^Y \partial_Y W \quad (2.39)$$

$$K^Z = 2\sqrt{6} \delta_{rs} Q^r R^{sXZ} \partial_X W \quad (2.40)$$

$$|q'|^2 = \frac{3}{2} |K|^2 g^2 e^{2w} (1 - \Lambda^2) \quad (2.41)$$

$$K_Z = \sqrt{6} W R^{rX}{}_Z D_X Q_r \quad (2.42)$$

$$|\partial W|^2 = \frac{|K|^2}{6} \quad (2.43)$$

$$3W q'^X \partial_X W D_Z Q_t = |K|^2 Q^r q'^X R_{ZX}^s \epsilon_{srt} \quad (2.44)$$

2.3 Matter field conditions: Gaugini equation

Next let us consider gaugini: using (2.16) to replace $\gamma_0 \epsilon$ in (2.6) one easily obtains

$$\Lambda \phi'^x + \sqrt{\frac{3}{2}} h_I^x (v'a^I + a'^I) = 0 \quad (2.45)$$

$$2g\Lambda h^{xI} P_I^s - \sqrt{\frac{3}{2}} e^{-w} h_I^x (v'a^I + a'^I) f^s = 0 \quad (2.46)$$

which gives

$$\phi'^x f^s = -2ge^w h^{xI} P_I^s \quad (2.47)$$

and using

$$h_x^I P_I^s = -\frac{3}{2} \partial_x (W Q^s) \quad (2.48)$$

one finally has

$$\partial_x Q^s = 0 \quad (2.49)$$

$$\pm\sqrt{1-\Lambda^2}\phi'^x = 3ge^w g^{xy}\partial_y W \quad (2.50)$$

$$\sqrt{6}g\Lambda\partial_x W = \mp e^{-w}h_I^x(v'a^I + a'^I)\sqrt{1-\Lambda^2} \quad (2.51)$$

We can rewrite the information on the scalars in a compact way defining

$$\varphi^\Sigma = \begin{cases} \phi^x & \text{for } \Sigma = 1, \dots, n_V \\ q^X & \text{for } \Sigma = n_V + 1, \dots, n_V + 4n_H \end{cases}$$

as

$$\tilde{D}_r Q^s = 0 \quad (2.52)$$

$$\varphi'^\Lambda = \pm 3ge^w (1-\Lambda^2)^{\frac{1}{2}\Delta} g^{\Lambda\Sigma}\partial_\Sigma W \quad (2.53)$$

with

$$(2.54)$$

$$\Delta = \begin{cases} -1 & \text{for } \Lambda = 1, \dots, n_V \\ 1 & \text{for } \Lambda = n_V + 1, \dots, n_V + 4n_H \end{cases} \quad (2.55)$$

where $g^{\Lambda\Sigma}$ is simply the product metric.

Let us discuss the consequences of the above relations. They are a generalization of the ones obtained in [1]. First of all a strong similarity with the domain wall case [2] emerges again: this observation is non trivial because the two configurations are quite different and it suggests that it should be possible to obtain a very general insight on BPS solutions in presence of generic matter couplings. To be more specific in the two situations it happens that the phase of prepotential Q^r do not depend on the vector multiplet scalars: under this condition the potential $\mathcal{V}(q, \phi)$ reduces the form that has been put forward for gravitational stability

$$\mathcal{V} = -6W^2 + \frac{9}{2}g^{\Lambda\Sigma}\partial_\Lambda W\partial_\Sigma W \quad (2.56)$$

It is easy to see that in this case critical points of W are also critical points of \mathcal{V} . Furthermore we find that $\varphi^\Lambda \propto \partial_\Lambda W$ but now the gauge interaction distinguishes between charged q^X and uncharged ϕ^x via the factor $1-\Lambda^2$. At the end we want to underline the importance of (2.45) that practically gives the component of the field strength on h_x^I and with (2.64) determines it as a vector of special geometry. This information will be crucial to check whether BPS solutions satisfy the equations of motion.

2.4 Further restrictions

As usual we have to compare the previous information with the one coming from the other integrability conditions. Let us consider equation (2.12): it is easy to show that or all the coefficients vanish or it must be equivalent to (2.16). The first case reduces to the case in which all the coefficients of (2.14) vanish. The second case occurs when the following conditions are true:

$$f^0 = -\frac{v' - g^2 r e^{2w} W^2}{v' a + a'} \quad (2.57)$$

$$f^r = -g r e^w W Q^r \quad (2.58)$$

with $a^I P_I^r$ parallel to $h^J P_J^r$

$$a^I P_I^r = \beta(r) h^J P_J^r \quad (2.59)$$

for some function β .

From the properties of very special geometry the modulus of vector h_I can be normalized to one $h_I h^I = 1$, so that the set (h^I, h_x^I) is a base for the $n_V + 1$ -dimensional space with $h_I h_x^I = 0$. Then the following relation holds

$$a^I = a h^I + l^x h_x^I \quad (2.60)$$

Using the above decomposition in (2.28) and (2.59) we get

$$\begin{cases} (a - \Lambda) D_Z P^r = -l^x D_Z P_x^r \\ (a - \beta) P^r = -l^x P_x^r \end{cases} \quad (2.61)$$

that taking in account the BPS demand $\partial_x Q^r = 0$ gives

$$\begin{cases} \beta = \Lambda \\ a - \Lambda = \sqrt{\frac{3}{2}} l^x \partial_x \ln W = -\frac{r}{\sqrt{6}} l_x \phi'^x \\ a - \Lambda = \sqrt{\frac{3}{2}} l^x \partial_x \ln \partial_Z W \end{cases} \quad (2.62)$$

We continue to derive the other equations from integrability conditions. Equation(2.57) together with (2.18) gives

$$1 + 2g^2 r^2 W^2 + \frac{r^2 e^{-2w}}{4} [v'^2 - (v' a + a')^2] = e^{-2w} (1 + \frac{r}{2} v')^2 \quad (2.63)$$

If we substitute (2.34) into (2.57)

$$\Lambda = \frac{r v' - 1 + \Lambda^2}{r(v' a + a')} \quad (2.64)$$

Using (2.16) in (2.13) we obtain the equations

$$gr(v'a + a')W + gr\Lambda W' \mp \frac{1}{2}\partial_r [re^{-w}(v'a + a')] \sqrt{1 - \Lambda^2} = 0 \quad (2.65)$$

$$\begin{aligned} & \mp gr(v'a + a')W\sqrt{1 - \Lambda^2} + \Lambda\partial_r(e^{-w}) - \Lambda g^2 r e^w W^2 \\ & + \frac{1}{2}\partial_r [re^{-w}(v'a + a')] = 0 \end{aligned} \quad (2.66)$$

Similarly from (2.11) we have

$$\begin{aligned} & \Lambda g \tilde{D}_r \left(\sqrt{\frac{3}{2}} e^v a^I P_I^s \right) + \frac{g}{2} e^v W' Q^s \mp \sqrt{1 - \Lambda^2} Q^s \left[\frac{1}{2} g^2 e^{v+w} W^2 + \frac{1}{2} e^{v-w} (v'a + a')^2 \right. \\ & \left. - \frac{1}{2} \partial_r (v' e^{v-w}) \right] = 0 \end{aligned} \quad (2.67)$$

$$\begin{aligned} & \mp \sqrt{1 - \Lambda^2} g e^v W' + \Lambda e^v \partial_r [e^{-w} (v'a + a')] + g^2 e^{v+w} W^2 + e^{v-w} (v'a + a')^2 \\ & - \partial_r (v' e^{v-w}) = 0 \end{aligned} \quad (2.68)$$

Note that all the relations already derived reduce to those in [1] if we take $\Lambda = a$. So it is easy to conclude that the particular case $l^x \equiv 0$ is compatible with the BPS conditions and reduces to the set of equations (5.1)-(5.4) of [1] plus the one for the scalars of the vector multiplets.

3 Static BPS configurations

In this section we derive the independent set of equations that characterizes BPS configurations.

We start by considering integrability conditions (2.65) and (2.66): subtracting from (2.65) the equation (2.66) multiplied by $\mp\sqrt{1 - \Lambda^2}$ we get

$$gr\Lambda(v'a + a')W + grW' \mp \sqrt{1 - \Lambda^2} w' e^{-w} \mp g^2 r \sqrt{1 - \Lambda^2} e^w W^2 = 0 \quad (3.1)$$

Using (2.64) and (2.34) it gives

$$W' = -(v' + w')W \quad (3.2)$$

that implies $gr_0 W = \mp e^{-(v+w)}$ where r_0 is a constant. This last expression can be rewritten considering again (2.34) as

$$e^v = \frac{r}{r_0 \sqrt{1 - \Lambda^2}} \quad (3.3)$$

which is fundamental to demonstrate the compatibility of the BPS conditions. Indeed taking the derivative with respect to r and comparing with (2.64) we obtain

$$v'a + a' = v'\Lambda + \Lambda' \quad (3.4)$$

This means that the integrability conditions and consequently the BPS equations for the metric (w and v) and for the scalars of the hypermultiplets have the same form as the ones in [1]: hence their consistency is ensured and the only change is the replacement of a by Λ . The new ingredients here, due to the introduction of vector multiplets, are then the equation for ϕ'^x and the relations between a , l^x and Λ (2.62).

Note that here a' is not the derivative of a with respect to r . Indeed we have defined $a' \equiv h_I a'^I$. Using (2.60) it is easy to find

$$a' = \partial_r a - \sqrt{\frac{2}{3}} l_x \phi'^x \quad (3.5)$$

Substituting it in (3.4) and using the second eq. of (2.62) we get

$$v'a + \partial_r a + \frac{2}{r}(a - \Lambda) = v'\Lambda + \Lambda' \quad (3.6)$$

which gives

$$\Lambda = a + \frac{\mu}{r^2} e^{-v} \quad (3.7)$$

where μ is an integration constant. According to (2.62) the last expression can be rewritten to show in a transparent manner the relation between μ and l^x as

$$\frac{\sqrt{6}\mu}{r^3} e^{-v} = l_x \phi'^x \quad (3.8)$$

The implication of this expression on the existence of fixed points still has to be clarified. Now it is not so difficult to show that the BPS conditions we have derived satisfy the equations of motion. We refer the reader to the appendix B for the technical details.

To summarize what it has been obtained, we conclude this section presenting a set of independent BPS equations

$$1 = \Lambda^2 e^{-2w} \left[1 + \frac{r}{2} \left(v' + \frac{\Lambda'}{\Lambda} \right) \right]^2 \quad (3.9)$$

$$e^v = \frac{r}{r_0 \sqrt{1 - \Lambda^2}} \quad (3.10)$$

$$\sqrt{1 - \Lambda^2} = \mp g e^w r W \quad (3.11)$$

$$q'^Z = \pm 3g e^w \sqrt{1 - \Lambda^2} \partial^Z W \quad (3.12)$$

$$\phi'^x = \pm 3g e^w \frac{1}{\sqrt{1 - \Lambda^2}} \partial^x W \quad (3.13)$$

$$\Lambda = a + \frac{\mu}{r^2} e^{-v} \quad (3.14)$$

$$\frac{\sqrt{6}\mu}{r^3} e^{-v} = l_x \phi'^x \quad (3.15)$$

$$\Lambda \phi'^x + \sqrt{\frac{3}{2}} h_I^x (v' a^I + a'^I) = 0 \quad (3.16)$$

We note that the last four equations are the new ones with respect to [1] due to the presence of vector multiplets.

As an application we can immediately determine the fixed point solutions of these equations. Strictly speaking they are the solutions having constant scalars, that are defined by the conditions $\phi'^x = 0$ and $q'^X = 0$. However one can include also the asymptotic fixed point solutions, which are characterized by $\phi'^x \rightarrow 0$ and $q'^X \rightarrow 0$ for some special values of r .

For now we just consider the first case. First of all one finds that the fixed point solutions correspond to the stationary point solutions of the potential W :

$$K_Z = \partial_Z W = \partial_z W = 0 . \quad (3.17)$$

giving a fixed value $W \neq 0$.

Furthermore (3.15) requires $\mu = 0$ so that $\Lambda = a$ whereas Q^r must be covariantly constant and finally the relation $h^{Ix} P_I^s = 0$ must be true.

The resulting configuration is then

$$e^v = \frac{\gamma + r^2}{r^2} \delta \sqrt{1 + \frac{g^2 W^2 r^6}{(\gamma + r^2)^2}} \quad (3.18)$$

$$e^w = \frac{r^2}{\gamma + r^2} \frac{1}{\sqrt{1 + \frac{g^2 W^2 r^6}{(\gamma + r^2)^2}}} \quad (3.19)$$

$$F_{rt}^I = -\sqrt{6} \frac{\gamma \delta}{r^3} h^I \quad (3.20)$$

where the two integration constants γ and δ are related to the electric charges Q^I by

$$Q^I = \sqrt{6} \gamma \delta h^I . \quad (3.21)$$

We have derived these solutions (as we have done for the BPS conditions (3.9-3.16)) assuming $0 < \Lambda^2 < 1$, it remains to study the two singular case $\Lambda^2 = 1$ and $\Lambda = 0$.⁴ It is quite easy to show that the last case doesn't correspond to any solution. For $\Lambda^2 = 1$ the BPS equations do not imply the equation of motion because some of them become singular (and consequently the demonstration given in the appendix B does not hold). However it is sufficient to impose by hand the Maxwell equations. The resulting configuration is

$$W = P_I = 0 \tag{3.22}$$

$$e^v = c \left(1 - \frac{b}{2r^2} \right) \tag{3.23}$$

$$e^w = \frac{1}{1 - \frac{b}{2r^2}} \tag{3.24}$$

$$F_{rt}^I = \pm \sqrt{\frac{3}{2}} \frac{bc}{r^3} h^I \tag{3.25}$$

This configuration corresponds to the Reissner-Nordström (extreme) black-hole of the minimal gauged theory and can be obtained from the general case as a limit $W \rightarrow 0$. Actually from the comparison of the two cases it seems that the presence of a nonvanishing W “regularizes” the horizon which disappears.

At this point one should analyze the class of asymptotically fixed point solutions which can be obtained perturbing the configurations just found. This requires a much more subtle investigation.

4 Towards an explicit solution

As we have already argued in the introduction the most interesting solutions of the form (2.8) are the ones which correspond to asymptotically AdS extreme black holes. However to find solutions of this kind solving the BPS equations (3.9-3.16) for an explicit model is quite hard. To understand better the origins of these difficulties, let us consider the other (few) classes of BPS solutions with hypermultiplet and vector multiplet couplings turned on already existing in the literature. Also for very simple models it seems always necessary to use a numerical approach. So it can be easily supposed that the numerical treatment will be the only possibility to perform explicit solutions. For example this happens in the flat domains wall case [2]. Although there are some similarities that we have already discussed, our case is much more complicated by the presence of Λ . Indeed our equations become almost of that form only for $\Lambda' = 0$. But this choice is too restrictive because it fixes completely the metric and, unlikely, to a form that does not have the nice features we are looking for [1].

⁴It is immediate to see that no solutions with non trivial scalars exist for $|\Lambda| = 1, 0$, so the present analysis covers all the possible configurations of this kind.

In addition, we have to satisfy (3.13-3.16) which, as we will discuss later on, can be seen as a sort of consistency constraint. These ones together with presence of Λ are a nontrivial obstruction to the application of a numerical method without assuming any ansatz on the form of Λ and to distinguish the true solutions from the artifacts.

So let us discuss the general features of eq. (3.13-3.16) focusing in particular on the meaning of the last three equations. Indeed these give the field strength F^I in terms of ϕ'^x and Λ :

$$F_{rt}^I \propto v' a^I + (a^I)' = (v' \Lambda + \Lambda') h^I - \sqrt{\frac{2}{3}} \Lambda \phi'^x h_x^I \quad (4.1)$$

Now we have to compare this expression with the decomposition for a^I (2.60) and with (2.62). Following the analysis in the section 3 we obtain the relations regarding the h^I projection that are exactly the (3.14) and (3.15). At this point we have to study the consequences of the h_x^I projection:

$$\partial_r l^x + l^z \phi'^y B_{zy}^x - \sqrt{\frac{2}{3}} (a - \Lambda) \phi'^x + v' l^x = 0 \quad (4.2)$$

where we define $B_{zy}^x = B_{yz}^x \equiv (\partial_y h_z^J) h_J^x$. Together with

$$a - \Lambda = -\frac{r}{\sqrt{6}} l_x \phi'^x = -\frac{\mu}{r^2} e^{-v} \quad (4.3)$$

(4.2) imposes a non trivial constraint that the solution must satisfy. To clarify better the meaning of this statement let us consider the specific case of one vector multiplet with a generic number of hypermultiplets present (in fact this analysis can carry out for any other specific model in which B_{zy}^x is known explicitly). Using the parametrization in [2] we have

$$B_{\rho\rho}^\rho = -\frac{3}{2} \frac{(\partial_\rho^2 h^J) \partial_\rho h_J}{g_{\rho\rho}} = \frac{1}{\rho} \quad (4.4)$$

that gives

$$l = l_0 e^{v(r_0)} \frac{\rho(r_0)}{\rho(r)} \exp \left[-4 \int_{r_0}^r dr r \left(\frac{\rho'}{\rho} \right)^2 \right] e^{-v} \quad (4.5)$$

$$\frac{\rho'}{\rho^3} \exp \left[-4 \int_{r_0}^r dr r \left(\frac{\rho'}{\rho} \right)^2 \right] = \frac{C}{r^3} \quad (4.6)$$

where $C = \frac{\sqrt{6} \mu e^{-v(r_0)}}{12 l_0 \rho(r_0)}$. For example, if we suppose for ρ a power-law behavior, $\rho = \alpha r^\beta$, the above relation fixes β to be or -1 or $1/2$ that rules out a lot of possible solutions. Indeed

it is possible to show that these are the only non trivial solutions of (4.6). To see this let us consider the derivative of (4.6): this condition reduces to an ordinary differential equation

$$y' = 4ry^3 + 2y^2 - \frac{3}{r}y \quad (4.7)$$

where y is the logarithmic derivative of ρ , $y \equiv \frac{\rho'}{\rho}$. The above equation can be expressed in a convenient form (to be easily integrated by separation of variables) in terms of $t \equiv y/z$ where $z = 1/r$:

$$\frac{dt}{dz} = -\frac{4}{z}t(t+1)(t-\frac{1}{2}) \quad (4.8)$$

One recognizes immediately in the three constant solutions of (4.8), $t = 0$, $t = -1$, $t = 1/2$, respectively the trivial solution $\rho' = 0$ of (4.6) and $\rho = \alpha r^\beta$, $\beta = -1, 1/2$. The non constant solutions live in the four regions delimited by the constant ones and they are defined by

$$\left| \frac{t}{t_0} \right|^{1/2} \left| \frac{t+1}{t_0+1} \right|^{-1/6} \left| \frac{t-1/2}{t_0-1/2} \right|^{-1/3} = \frac{z}{z_0} \quad (4.9)$$

At the end all the problem reduces to compute the real roots of a third-order polynomial

$$(1-b)t^3 + \frac{3}{4}b t - \frac{b}{4} = 0 \quad (4.10)$$

where $b = (cz)^6$, hence greater than zero, for $t > 0 \vee t < -1$ while $b = -(cz)^6$ elsewhere. c is a positive constant of integration. Performing explicit calculations, one obtains that all the solutions for $t > 0$ ($t < 0$) behave asymptotically for $z \rightarrow \infty$ like $t = 1/2$ ($t = -1$). The solutions in the regions $t < -1$ and $t > 1/2$ exist only for a finite range in z ,⁵ $z > 1/c$, while the solutions in the intermediate regions $0 < t < 1/2$ and $-1 < t < 0$ interpolate between $t = 0$ and respectively $t = 1/2$ and $t = -1$. At this point one has to check which of the solutions of (4.8) satisfy also (4.6). It can be easily shown, for example considering the behavior for $z \simeq 0$, that only the solutions with t constant survive.

Let us stress the relevance of the condition derived in this section. First of all the appearance of such strong requirement on the shape of the scalars of vector multiplets is quite surprising and, up to our knowledge, new. In particular it seems quite striking that for $n_V = 1$ the form of ρ is fixed a priori for any number of hypermultiplets and for any choice of the gauging. Indeed the implications of the above result on the construction of electro-static spherically symmetric solutions are quite severe. As first consequence one

⁵Thus we can discard this kind of solutions immediately.

aspects that a solution of this kind *can not exist* for a generic selection of the isometries. Indeed this is exactly the case: it is easy to check that for $n_H = 1$ for any combination of the killing vector of the form⁶

$$K = h^0 K_0 + h^1 K_1 \quad (4.11)$$

$$K_0 = \alpha k_{(1)} + \beta k_{(2)} \quad K_1 = \gamma k_{(1)} + k_{(2)} \quad (4.12)$$

with α, β, γ constant parameters. Due to the structure of the differential equations for the scalars the same (non) result holds also substituting $k_{(2)}$ with $k_{(3)}$ in (4.12).

As second consequence, closely related to the first, it does not exist a general criteria to determine which are the right choices for the gauging (in the sense that produce a solution) without an explicit try. Indeed, the general procedure is to evaluate the equation (3.13) for ρ , for the chosen prepotential and imposing the constraint. In this way one obtains an *algebraic* equation that the hypermultiplet scalars have to satisfy. This way of acting is quite laborious and imposes a strong limitation on the number of models that it can test. It could be nice to understand what happens in the presence of a generic number of vector multiplets. One expects that with more scalars the requirement will be in some sense relaxed. Anyway the system should be also in this case overconstrained.

5 Discussion

In this section we want to recall the results already obtained and to point out which topics deserve more study. First of all we have derived BPS equations, studying in the line of [1], the relations from the hyperini and the gaugini and the integrability conditions for the gravitini. We observe that the former ones have the same structure manifested in the domain wall case [2]: this suggests the possibility of determining some properties of BPS solutions without starting from the specific ansatz. The importance of a similar study is evident: for example this could permit us to give a definitive answer in the quest for a realistic cosmological model in gauged supergravity.

At the same time we have discovered a quite unexpected condition for the scalars of vector multiplets. As it emerges quite clearly from the last section, the analysis and the better understanding of this constraint is crucial for the construction of non trivial solutions. Let us stress again that for $n_V = 1$ the relations (4.5), (4.6) are sufficient to determine the space-time dependence of the vector multiplet scalar ρ independently by the choice of the prepotential. This last observation suggests that it could be possible to give an interpretations to this phenomena in terms of the six dimensional gauged supergravity where the scalars of the vector multiplets are just a component of the gauge fields. Another interesting question that arises quite naturally is whether this kind of relations is peculiar to this particular case or instead is a feature common to a larger class of charged solutions. This points are currently under investigation.

⁶We follow the notation and the parametrization of [2] for the metric and the isometries of the universal hypermultiplet, see the appendix A.

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A Conventions

In this appendix we present some definitions and properties that we use in our work. With

$$q^X \quad X = 1, \dots, 4n_H \quad (\text{A.1})$$

we denote the scalars of the hypermultiplets which are the coordinates of a quaternionic manifold. We introduce the $4n_H$ beins as

$$f_X^{iA}(q^Y), \quad i = 1, 2 \in SU(2), A = 1, \dots, 2n_H \in Sp(2n_H) \quad (\text{A.2})$$

The splitting of the flat indices in i and A reflects the factorization of the holonomy group in $USp(2)(\simeq SU(2)) \otimes USp(2n_H)$ which is the main feature of those spaces. The indices as a consequence of the symplectic structure are highered and lowered with the antisymmetric matrices

$$\epsilon_{ij}, \quad \mathbb{C}_{AB} \quad (\text{A.3})$$

$$\epsilon_{ij} = \epsilon^{ij}, \quad \epsilon_{12} = 1 \quad (\text{A.4})$$

$$\mathbb{C}_{AB}\mathbb{C}^{CB} = \delta_A^C, \quad \mathbb{C}^{AB} = (\mathbb{C}_{AB})^*. \quad (\text{A.5})$$

following the NW-SE convention [2].

The important relation

$$f_{XiC}f_{Yj}^C = \frac{1}{2}\epsilon_{ij}g_{XY} + R_{XYij} \quad (\text{A.6})$$

can be viewed as a definition for the quaternionic metric g_{XY} and for the $SU(2)$ curvature R_{XYij} .

We use the symbols p_{Xi}^j for the $SU(2)$ spin connection whereas ω_μ^{ab} denotes the usual Lorentz spin connection. The covariant derivative which appears in the gravitini supersymmetry variation acts on the symplectic Majorana spinors ϵ_i as

$$\mathcal{D}_\mu \epsilon_i = \partial_\mu \epsilon_i + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon_i - \partial_\mu q^X p_{Xi}^j \epsilon_j - g A_\mu^I P_{Ii}^j \epsilon_j \quad (\text{A.7})$$

where the generalized spin connection receives the following contributions: the first term represents the Lorentz action while the others can be identified with the $SU(2)$ action plus a term due to the $SU(2)$ R-symmetry gauging. A_μ^I are $(n_V + 1)$ 1-forms and P_I^r are the prepotentials while g is the gauge coupling. We adopt the convention to define for the quantities with an I index the corresponding “dressed” ones like $P^r \equiv P_I^r h^I$ or $F^{\mu\nu} \equiv F_I^{\mu\nu} h^I$. We note that in this notation the subcase $n_V = 0$ is recovered in a natural way being $I = 0$ and $h^I = h^0 = 1$.⁷

⁷This implies, from the definition of very special geometry, that the normalization of C_{IJK} is given by $C_{000} = 1$.

It is useful to introduce the projection on the Pauli matrices for quantities in the adjoint representation of $SU(2)$, for example

$$R_{XYi}{}^j = R_{XY}^r (i\sigma_r)_i{}^j \quad (\text{A.8})$$

where $(\sigma_r)_i{}^j$ are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.9})$$

which satisfy

$$(\sigma_r)_i{}^j (\sigma_s)_j{}^k = \delta_{rs} \delta_i{}^k + i\epsilon_{rst} (\sigma_t)_i{}^k \quad (\text{A.10})$$

$$[\sigma_r, \sigma_s] = 2i\epsilon_{rst} \sigma^t \quad (\text{A.11})$$

The prepotentials are defined by the relation

$$R_{XY}^r K^Y = D_X P^r \quad (\text{A.12})$$

$$D_X P^r := \partial_X P^r + 2\epsilon^{rst} p_X^s P^t \quad (\text{A.13})$$

where D_X is the $SU(2)$ covariant derivative. They can be expressed in terms of the Killing vectors

$$P^r = \frac{1}{2n_H} D_X K_Y R^{XY} \quad (\text{A.14})$$

The scalar potential can be expressed for a generic number of hypermultiplets and vector multiplets as

$$\mathcal{V} = g^2 [-P_r P^r + 2P_{xr} P_y^r g^{xy} + 2N_{Ai} N^{Ai}] \quad (\text{A.15})$$

with

$$N^{Ai} = \frac{\sqrt{6}}{4} h^I K_I^X f_X^{Ai} = \frac{2}{\sqrt{6}} f_X^{Ai} R^{YX} D_Y P^r, \quad (\text{A.16})$$

$$P_x^r \equiv -\sqrt{\frac{3}{2}} \partial_x P^r = h_x^I P_I^r \quad (\text{A.17})$$

Defining the superpotential W by $P^r = \sqrt{\frac{3}{2}} W Q^r$ with $Q^r Q_r = 1$ the potential becomes

$$\mathcal{V} = -6g^2 W^2 + \frac{9}{2} g^2 [g^{\Lambda\Sigma} \partial_\Lambda W \partial_\Sigma W + W^2 g^{xy} \partial_x Q^r \partial_y Q_r] \quad (\text{A.18})$$

where Λ is the curl index of the entire $n_V + 4n_H$ -dimensional scalar manifold. From the above relation it follows that the requirement on \mathcal{V} to be of the form $\mathcal{V} = -6g^2 W^2 + \frac{9}{2} g^2 [g^{\Lambda\Sigma} \partial_\Lambda W \partial_\Sigma W]$, which ensures the gravitational stability, is

$$\partial_x Q^r = 0$$

as found in the sect. 2.3.

The universal hypermultiplet ($n_H = 1$) corresponds to the quaternionic Kähler space $\frac{SU(2,1)}{SU(2) \times U(1)}$. A significant parametrization, from a M-theory point of view, is [2]

$$q^X = \{V, \sigma, \theta, \tau\}$$

with the metric

$$ds^2 = \frac{dV^2}{2V^2} + \frac{1}{2V^2} (d\sigma + 2\theta d\tau - 2\tau d\theta)^2 + \frac{2}{V} (d\tau^2 + d\theta^2) . \quad (\text{A.19})$$

Using the general properties of quaternionic geometry it is possible from (A.19) to derive explicitly all the quantities presented above, in particular the Killing vectors and the prepotentials of the eight isometries of manifold. For the axionic shift we have:

$$\vec{k}_{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{P}_{(1)} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4V} \end{pmatrix} \quad (\text{A.20})$$

For $k_{(2)}$ and $k_{(3)}$ we have

$$\vec{k}_{(2)} = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \end{pmatrix} \quad \vec{P}_{(2)} = \begin{pmatrix} -\frac{1}{\sqrt{V}} \\ 0 \\ -\frac{\theta}{V} \end{pmatrix} \quad (\text{A.21})$$

$$\vec{k}_{(3)} = \begin{pmatrix} 0 \\ -2\tau \\ 1 \\ 0 \end{pmatrix} \quad \vec{P}_{(3)} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{V}} \\ \frac{\tau}{V} \end{pmatrix} \quad (\text{A.22})$$

B Equations of motion

The equations of motion of the lagrangian (2.4) in the presence of hypermultiplets and vector multiplets are

$$- \mathcal{R}_{\mu\nu} + a_{IJ} F_{\mu a}^I F_{\nu}^{Ja} + g_{XY} D_{\mu} q^X D_{\nu} q^Y + g_{xy} \partial_{\mu} \phi^x \partial_{\nu} \phi^y - \frac{1}{6} |F|^2 g_{\mu\nu} + \frac{2}{3} \mathcal{V} g_{\mu\nu} = 0 \quad (\text{B.1})$$

from which it follows in particular

$$- \frac{3}{2} R + \frac{1}{4} |F|^2 + \frac{3}{2} g_{XY} D_{\mu} q^X D^{\mu} q^Y + \frac{3}{2} g_{xy} \partial_{\mu} \phi^x \partial^{\mu} \phi^y - \frac{9}{4} |a^I K_I|^2 + 5\mathcal{V} = 0 \quad (\text{B.2})$$

The variation with respect to the gauge fields gives

$$\mathcal{D}_a(a_{IK}F^{Kae}) + \frac{1}{2\sqrt{6}}C_{IJK}\epsilon^{abcde}F_{ab}^JF_{cd}^K - gK_I^X D^e q^Y g_{XY} = 0 \quad (\text{B.3})$$

Finally the equations for the scalars are

$$\hat{D}_\mu D^\mu q^W + gA^{\mu I} D_\mu K_I^W = g^{WX} \partial_X \mathcal{V} \quad (\text{B.4})$$

$$\hat{D}_\mu D^\mu \phi^x + gA^{\mu I} D_\mu K_I^x = g^{xy} \partial_y \mathcal{V} + \frac{1}{4} g^{xy} \partial_y a_{IJ} F_{\mu\nu}^I F^{J\mu\nu} \quad (\text{B.5})$$

Here \mathcal{D} is the covariant derivative with respect to the spin connection and \hat{D} is a totally covariant derivative, ie with respect to all the indices. So for example

$$\hat{D}_\mu D^\mu \varphi^\Lambda = \mathcal{D}_\mu D^\mu \varphi^\Lambda + \Gamma_{\Sigma\Theta}^\Lambda D_\mu \varphi^\Sigma D^\mu \varphi^\Theta$$

and in general

$$D_\mu f^*(q, \phi) = D_\mu q^X \partial_X f^* + D_\mu \phi^x \partial_x f^* \quad (\text{B.6})$$

Now we specialize the above relations to the problem studied in this work. Due to symmetry of the class of solutions considered only the Einstein equations for the components (tt) , (rr) and $(\theta\theta)$ are independent:

$$\begin{aligned} -e^{v-w} \partial_r (v' e^{v-w}) - 3 \frac{v'}{r} e^{2(v-w)} + e^{2(v-w)} (v' \Lambda + \Lambda')^2 + 4g^2 e^{2v} W^2 \\ - 3g^2 e^{2v} (1 - 3\Lambda^2) \left[\frac{1}{1 - \Lambda^2} g^{xy} \partial_x W \partial_y W + g^{XY} \partial_X W \partial_Y W \right] = 0 \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} e^{w-v} \partial_r (v' e^{v-w}) - \frac{3}{r} w' - (v' \Lambda + \Lambda')^2 - 4g^2 e^{2w} W^2 \\ + 3g^2 e^{2w} (1 + 3(1 - \Lambda^2)) \left[\frac{1}{1 - \Lambda^2} g^{xy} \partial_x W \partial_y W + g^{XY} \partial_X W \partial_Y W \right] = 0 \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} r e^{-2w} (v' - w') - 2(1 - e^{-2w}) + \frac{1}{2} r^2 e^{-2w} (v' \Lambda + \Lambda')^2 - 4g^2 r^2 W^2 \\ + 3g^2 r^2 \left[\frac{1}{1 - \Lambda^2} g^{xy} \partial_x W \partial_y W + g^{XY} \partial_X W \partial_Y W \right] = 0 \end{aligned} \quad (\text{B.9})$$

$$\sqrt{\frac{3}{2}} g e^v a^I K_{IX} q'^X = 0 \quad (\text{B.10})$$

where we use the BPS relations for ϕ'^x , q'^X and $|K|^2 = 6g^{XY} \partial_X W \partial_Y W$. Following the manipulations of [1] we consider the sum of (B.7) and (B.8) multiplied by $e^{2(v-w)}$

$$\frac{(v' + w')}{r} e^{-2w} = 3g^2 \left[\frac{1}{1 - \Lambda^2} g^{xy} \partial_x W \partial_y W + g^{XY} \partial_X W \partial_Y W \right] \quad (\text{B.11})$$

$$= \pm g \frac{e^{-w}}{\sqrt{1 - \Lambda^2}} W' \quad (\text{B.12})$$

The above expression is the direct generalization of the one in [1] and is identically satisfied by (2.34) and (3.2). Now by the substitution of (B.12) in (B.7), (B.8) and (B.9) it is easy to check that also these expressions are identically satisfied by the set of BPS equations. Finally (B.10) is solved by (2.27) and the equation $q'^Z K_Z = 0$, which follows for example from (2.29) and (2.40).

Next consider the equations for the gauge fields:

$$K_X^I q'^X = 0 \quad (\text{B.13})$$

$$\partial_r(a_{IJ}e^{-w}r^3(v'a^J + a'^J)) - e^w r^3 g^2 g_{XY} K_I^X K_J^Y a^J = 0 \quad (\text{B.14})$$

It is convenient to project these equations on the base (h_I, h_J^x) . The contraction of (B.13) with h_I gives $K_X q'^X = 0$ which we have already shown to be a consequence of BPS equations.

The contraction with h_{Ix} gives $q'^X K_X^I h_{Ix} = 0$ which by means of (2.29) is equivalent to $\partial^X W K_X^I h_{Ix} = 0$. But from (2.42) and (2.49) we have

$$\partial_x K^Z = \frac{\partial_x W}{W} K^Z \quad (\text{B.15})$$

so that

$$\partial^X W K_X^I h_{Ix} = \sqrt{\frac{3}{2}} \partial^X W \partial_x K_X = \sqrt{\frac{3}{2}} \frac{\partial_x W}{W} \partial^X W K_X = 0 \quad (\text{B.16})$$

After an integration by parts and using (3.4) the contraction of (B.14) with h^I gives

$$\partial_r[r^3 e^{-w}(v'\Lambda + \Lambda')] + \sqrt{\frac{2}{3}} r^3 e^{-w} \phi'^x h_{Ix} (v'a^I + a'^I) - e^w r^3 g^2 g_{XY} K^X K_J^Y a^J = 0 \quad (\text{B.17})$$

and using (2.51) becomes

$$\partial_r[r^3 e^{-w}(v'\Lambda + \Lambda')] \mp \frac{2gr^3 \Lambda W'}{\sqrt{1-\Lambda^2}} \pm \frac{2gr^3 \Lambda q'^X \partial_X W}{\sqrt{1-\Lambda^2}} - e^w r^3 g^2 g_{XY} K^X K_J^Y a^J = 0 \quad (\text{B.18})$$

The above equation, after the use of (2.27), can be easily related to the computations in [1] with Λ in place of a .

The contraction with h_y^I gives

$$\begin{aligned} r^3 e^{-w} (\Lambda' + v'\Lambda) & \sqrt{\frac{2}{3}} \phi'^x h_y^I h_{Ix} \mp h_y^I \partial_r h_I^x \frac{\partial_x W \sqrt{6g\Lambda r^3}}{\sqrt{1-\Lambda^2}} \\ & \mp \partial_r \left(\frac{\partial_y W \sqrt{6g\Lambda r^3}}{\sqrt{1-\Lambda^2}} \right) = -g^2 r^3 e^w \sqrt{\frac{3}{2}} \partial_y K^Z K_Z \end{aligned} \quad (\text{B.19})$$

From (2.34) and (2.50) we find

$$-rW\phi'^x = 3\partial^x W \quad (\text{B.20})$$

and from (2.33), (2.34)

$$g^2 e^{2w} r W = -2(W' - \phi'^x \partial_x W) \quad (\text{B.21})$$

Using these last equations together with (B.15) and (3.2) we have

$$h_y^I \partial_r h_I^x \frac{\partial_x W \Lambda r^2}{W} + \Lambda \partial_r \left(\frac{\partial_y W r^2}{W} \right) = -r^2 \Lambda \frac{\partial_y W}{W} \phi'^x \frac{\partial_x W}{W} \quad (\text{B.22})$$

By means of (2.49) and some integration by parts the following identity can be derived

$$\begin{aligned} h_y^I \partial_r h_I^x \partial_x W &= \sqrt{\frac{2}{3}} h_y^I \partial_r h_I^x \partial_x h^J P_J^s Q_s \\ &= -h_y^I \partial_r (h_I^x h_x^J) \frac{2}{3} P_J^s Q_s - \sqrt{\frac{2}{3}} \partial_r \partial_y h^J P_J^s Q_s \\ &= h_y^I \partial_r (h_I h^J) \frac{2}{3} P_J^s Q_s - \sqrt{\frac{2}{3}} \partial_r [\partial_y (P^s Q_s)] + \sqrt{\frac{2}{3}} q'^X \partial_X [\partial_y (P^s Q_s)] \\ &= \frac{2}{3} g_{yx} \phi'^x W - \partial_r (\partial_y W) + q'^X \partial_X (\partial_y W) \\ &= -2r \Lambda \frac{\partial_x W}{W} - \partial_r (\partial_y W) + q'^X \partial_X (\partial_y W) \end{aligned} \quad (\text{B.23})$$

where in the last step we have used (2.50) and (2.34). This together with (2.29) and (2.43) shows that (B.22) is identically satisfied.

The equations of motion for the hyperini are

$$\begin{aligned} e^{-(v+w)} r^{-3} \partial_r (r^3 e^{v-w} g_{ZY} q'^Y) - \frac{1}{2} q'^X \partial_X g_{ZY} q'^Y e^{-2w} + \frac{3}{4} g^2 \partial_Z g_{XY} a^I K_I^X a^J K_J^Y \\ + \frac{3}{2} g^2 g_{XY} a^I a^J \partial_Z K_I^X K_J^Y = g^2 \partial_Z \left(-6W^2 + \frac{3}{4} K^2 + \frac{9}{2} g^{xy} \partial_x W \partial_y W \right) \end{aligned} \quad (\text{B.24})$$

Using (2.27), (2.29), (2.43) and (2.50) it becomes

$$\begin{aligned} \pm e^{-w} 9 \frac{g}{r} \sqrt{1 - \Lambda^2} \partial_Z W &\pm 3(v' - w') g e^{-w} \sqrt{1 - \Lambda^2} \partial_Z W \\ &\pm e^{-2w} 3g \partial_Z W \partial_r (e^w \sqrt{1 - \Lambda^2}) = -12g^2 W \partial_Z W \end{aligned} \quad (\text{B.25})$$

which follows from (2.34) and the considerations in [1].

The equations of motion for the gaugini are

$$\frac{3}{4} \partial_x a_{IJ} e^{-2w} (a'^I + v' a^I) (a'^J + v' a^J) - \frac{1}{2} \partial_x g_{yz} \phi'^y \phi'^z e^{-2w} + r^{-3} e^{-(v+w)} \partial_r (r^3 e^{v-w} \phi'^y g_{xy})$$

$$-g^2 \partial_x \left(-6W^2 + \frac{3}{4}K^2 + \frac{9}{2}g^{zy} \partial_z W \partial_y W \right) = 0 \quad (\text{B.26})$$

From (2.50), (2.29) and (2.43) we find

$$\begin{aligned} r^{-3} e^{-(v+w)} \partial_r (r^3 e^{v-w} \phi'^y g_{xy}) &= \pm 9 \frac{g e^{-w} \partial_x W}{r \sqrt{1-\Lambda^2}} \pm 3g v' \frac{e^{-w} \partial_x W}{\sqrt{1-\Lambda^2}} \pm 3g e^{-w} \partial_x W \partial_r \frac{1}{\sqrt{1-\Lambda^2}} \\ &\quad + 3g \frac{e^{-w}}{\sqrt{1-\Lambda^2}} \left[\frac{3g e^w \partial_y \partial_x W \partial^y W}{\sqrt{1-\Lambda^2}} + 3g e^w \partial_x \partial_X W \partial^X W \sqrt{1-\Lambda^2} \right] \end{aligned} \quad (\text{B.27})$$

$$\begin{aligned} -g^2 \partial_x \left(-6W^2 + \frac{3}{4}K^2 + \frac{9}{2}g^{zy} \partial_z W \partial_y W \right) &= 12g^2 W \partial_x W - 9g^2 \partial_x \partial_Y W \partial^Y W \\ &\quad - \frac{9}{2}g^2 (\partial_x g^{yz} \partial_y W \partial_z W + 2\partial_x \partial_y W \partial^y W) \end{aligned} \quad (\text{B.28})$$

$$-\frac{1}{2} \partial_x g_{yz} \phi'^y \phi'^z e^{-2w} = -\frac{9}{2} g^2 \frac{\partial_y W \partial^y W}{\sqrt{1-\Lambda^2}} \quad (\text{B.29})$$

In a similar way as (B.23) one finds

$$\partial_x h_I^y \partial_y W = \frac{2}{3} W h_{Ix} + h_I \sqrt{\frac{2}{3}} \partial_x W - h_I^y \partial_x \partial_y W \quad (\text{B.30})$$

From this and (3.4), (2.51) and (2.64) we find

$$\begin{aligned} \frac{3}{4} \partial_x a_{IJ} e^{-2w} (a'^I + v' a^I) (a'^J + v' a^J) &= \mp \frac{6g e^{-w} \partial_x W}{\sqrt{1-\Lambda^2}} \left[v' - \frac{1-\Lambda^2}{r} \right] + \frac{6\Lambda^2 g^2 W \partial_x W}{1-\Lambda^2} \\ &\quad - \frac{9}{2} g^2 \frac{\Lambda^2}{1-\Lambda^2} (\partial_x \partial_y W \partial^y W - 2\partial_x g_{yz} \partial^y W \partial^z W) \end{aligned} \quad (\text{B.31})$$

Summing up all the terms

$$\begin{aligned} 0 &= \left(\pm 9 \frac{g}{r} e^{-w} \mp 3g e^{-w} v' \pm 3g e^{-w} \frac{\Lambda \Lambda'}{1-\Lambda^2} \right) \frac{\partial_x W}{\sqrt{1-\Lambda^2}} + 12g^2 W \partial_x W \\ &\quad + 6\Lambda^2 g^2 \frac{W \partial_x W}{1-\Lambda^2} \pm 6g e^{-w} \partial_x W \sqrt{1-\Lambda^2} \end{aligned} \quad (\text{B.32})$$

Using (2.34) to eliminate e^{-w} from the first and the last term and next (2.64) to eliminate $\Lambda' + \Lambda v'$, we finally see that the gaugini equations also are satisfied.

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